Chapter 3: Holographic Duality

$$\begin{array}{c}\text{quantum gravity} \\ \text{in } AdS_{d+1}\end{array} = CFT_d$$

Equivalence between two quantum systems

$\rightsquigarrow$ guess the dictionary

$\Rightarrow$ verify

$\Rightarrow$ make more guesses

- parameters, symmetries <u>should match</u>

e.g. $U(1)$ gauge $\leftrightarrow$ $U(1)$ global

3.1 General aspects

3.1.1. IR/UV connection

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \qquad (*)$$

$(*)$ is invariant under $(t, \vec{x}) \to \lambda (t, \vec{x})$
$z \to \lambda z$

$\Rightarrow$ extra dimension $\leftrightarrow$ scale

Note: $(t, \vec{x})$ defined in boundary units

$$d\tau = \frac{R}{z} dt \qquad dl = \frac{R}{z} dx$$

→ For some process of local energy E_{loc} and local length d_{loc} at z

$$d_{YM} = \frac{z}{R} d_{loc}, \quad E_{YM} = \frac{R}{z} E_{loc}$$

⇒ For the same process at different z:

(boundary) $z \to 0$: $E_{YM} \to \infty$, $d_{YM} \to 0$ (UV process)

$z \to \infty$: $E_{YM} \to 0$, $d_{YM} \to \infty$ (IR process)

⇒ typical bulk process, $E_{loc} \sim 1/R$

$$\Rightarrow \underline{E_{YM}} \sim 1/z$$

IR-UV connection

Wednesday Nov 21, notes from Haoyu Guo

From before:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^{\,2} + dz^2\right)$$

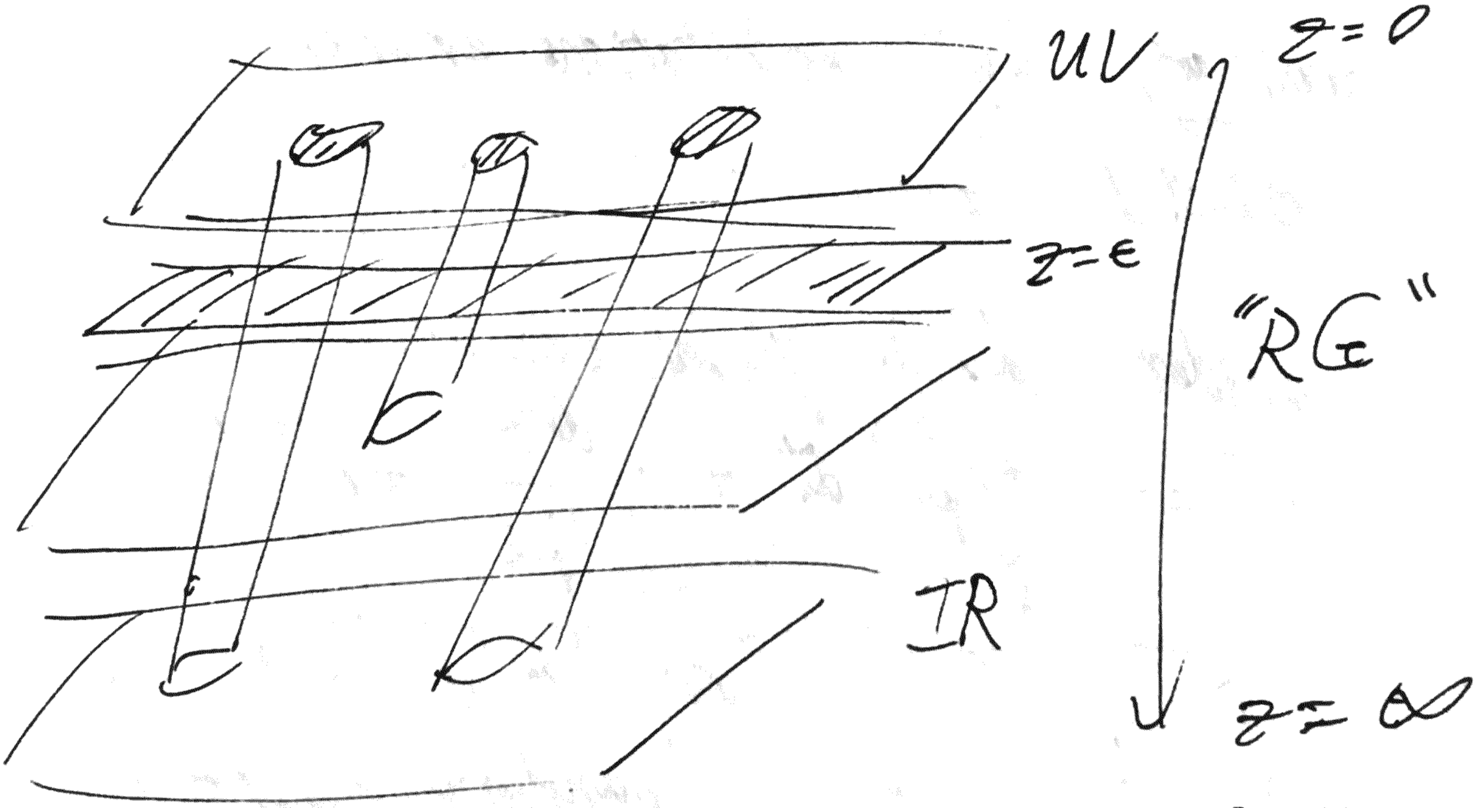
$$E_{YM} = \frac{R}{z} E_{loc}$$

$$d_{YM} = \frac{z}{R} d_{loc}$$

$$\cancel{E_{YM} \propto 1/z \qquad d_{YM} \propto z}$$

$$E_{loc} \sim 1/R, \quad d_{loc} \sim R$$

$$\Rightarrow \quad E_{YM} \propto \frac{1}{z}, \qquad d_{YM} \propto z$$



radial direction: geometrization of "scale"

IR-UV connection

Remarks:

1) $z \to 0$ $\quad E_{YM} \to \infty$

$d_{YM} \to 0$

put an IR-cutoff on gravity side at $z = \epsilon$

$\Rightarrow$ UV cutoff $\propto 1/\epsilon$ (energy)

distance $\propto \epsilon$

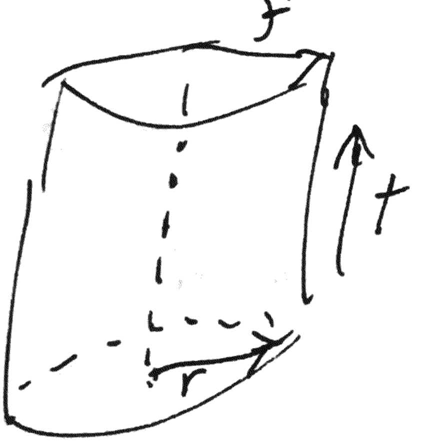
Check: basic idea of holographic principle (pset)

2) In a CFT on $R^{3,1}$ there exist <u>arbitrarily low</u> excitation energies.

reflected in $z \to \infty$ on the gravity side

3) Consider AdS in global coordinates:

$$ds^2 = -\underbrace{\left(1+\frac{r^2}{R^2}\right)}_{f} dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega_{d-1}^2$$



$S^{d-1} \times t$

on field theory side boundary sphere has radius R

$E_{YM} \sim 1/R$ (energy gap)

$$E_{YM} = f^{1/2} E_{loc}$$
$$= \left(1+\frac{r^2}{R^2}\right)^{1/2} E_{loc}$$
$$= \begin{cases} \infty & \text{as } r\to\infty \\ E_{loc} \sim 1/R & \text{as } r \to 0 \end{cases}$$

4) This works in more general asymptotic AdS metric

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2 d\Omega_{d-1}^2$$

away from boundary f decreases
$\Rightarrow E_{YM}$ decreases

3.1.2 Matching of the spectrum

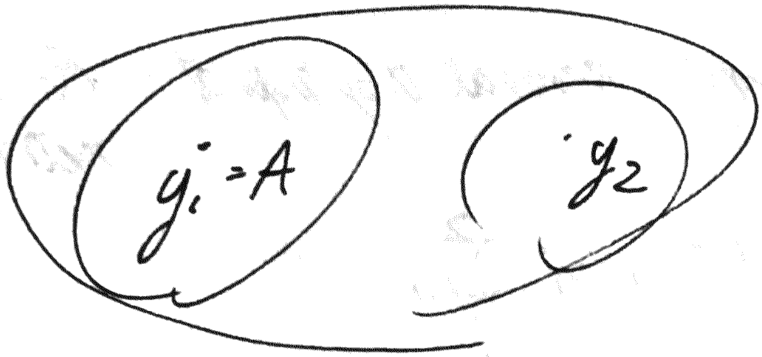
$$\underbrace{QG \text{ in } AdS_{d+1} = CFT_d}_{\text{same Hilbert space}}$$

physical states $\longleftrightarrow$ physical states

Classical gravity $\rightarrow$ Classical solutions (states)

geometry $\nrightarrow$ state

given a geometry: quantize matter fields $\Rightarrow$ a subset of quantum states on gravity side



boundary		gravity
$\lvert 0 \rangle$	$\longleftrightarrow$	pure AdS matter in vacuum
lowly excited states	$\longleftrightarrow$	lowly excited

group theory

boundary $\qquad\qquad$ gravity

$\mathcal{O}(x)$ $\longleftrightarrow$ $\varphi(x)$

conformal operators $\longleftrightarrow$ bulk fields

scalar operators $\longleftrightarrow$ scalar fields

J_μ $\longleftrightarrow$ A_μ

all quantum numbers of any symmetry should match

Lets use a simple scalar as an example to see how this mapping works

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Recall: in a matrix-type field theory, ~~key objects:~~

key objects: single-trace operators

$\langle \mathcal{O}\mathcal{O}\rangle \sim O(1)$ $\qquad$ $\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle_c \sim 1/N$

$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_c \sim 1/N^2$

leading order in large N:
Gaussian Theory

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on gravity side:

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\,\sqrt{-g}\,[R - 2\Lambda + \mathcal{L}_{matt}]$$

$$\mathcal{L}_{matt} = -\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 + O(\Phi^3) + \dots$$

$$g_{\mu\nu} = \underbrace{g^0_{\mu\nu}}_{\text{pure AdS}} + \kappa h_{\mu\nu}$$

$$16\pi G_N = 2\kappa^2$$

$$\Phi = 0 + \kappa\varphi$$

$$\Rightarrow S = \int d^{d+1}x\,\sqrt{-g}\,\Big[-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \kappa\varphi^3 + \kappa^2\varphi^4 + \dots$$
$$-(\partial h)^2 - \kappa h^3 + \kappa^2 h\varphi^2$$

$G_N \sim \frac{1}{N^2}$, $\kappa \sim \frac{1}{N}$

leading order in $1/N$ $\Rightarrow$ quadratic theory

$\rightsquigarrow$ (standard) quantization of φ

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eom for φ

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi\right)-m^2\varphi=0$$

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=\frac{R^2}{z^2}\left(-dt^2+d\vec{x}^2+dz^2\right)$$

$$\varphi(z,x^\mu)=\int\frac{d^dk}{(2\pi)^d}e^{ik\cdot x}\varphi(z,k)$$

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\varphi\right)-k^2z^2\varphi-m^2R^2\varphi=0$$

$$k^2=-\omega^2+\vec{k}^2$$

$$k^\mu=(\omega,\vec{k})$$

consider $z\to 0$

$$\rightsquigarrow z^2\partial_z^2\varphi+(1-d)z\partial_z\varphi-m^2R^2\varphi=0$$

let $\varphi\sim z^\alpha$

$$\alpha(\alpha-1)+(1-d)\alpha-m^2R^2=0$$

$$\Rightarrow\alpha=\frac{d}{2}\pm\underbrace{\sqrt{\frac{d^2}{4}+m^2R^2}}_{=:\nu}$$

$$\Delta\equiv\frac{d}{2}+\nu$$

$$\alpha_+=\Delta$$
$$\alpha_-=d-\Delta$$

$$\Rightarrow\varphi(k,z)=\underset{\sim}{A(k)}z^{d-\Delta}+\underset{\sim}{B(k)}z^{\Delta}$$

as $z\to 0$

Remarks:

(1) Exponents are real provided $m^2R^2 \geq -\frac{d^2}{4}$ $(*)$

one can show: (a) a theory is well-defined if $(*)$ is satisfied

(b) if $(*)$ is violated, there exist exponentially growing terms in time

$\Rightarrow$ instabilities

B.F. bound

contrast: in Mink

$$\partial^2\varphi - m^2\varphi = 0$$

$$\Rightarrow \omega^2 = \vec{k}^2 + m^2$$

$\underline{\underline{m^2<0}}$ $\quad \omega^2 < 0$ for $k=0$

$\rightsquigarrow \omega$ pure imaginary

$$\cancel{(2)\ |A|^2 = \frac{R^2}{z^2} g^{\mu\nu} A_\mu A_\nu}$$

In AdS, due to spacetime curvature,
the constant modes are not allowed

$\rightsquigarrow$ a field is forced to have some kinetic energy, compensating for some negative m^2

(2) AdS has a boundary, and light rays reach this boundary in finite time.

$\Rightarrow$ Energy can be exchanged at the boundary.

$\rightsquigarrow$ need to impose appropriate boundary conditions.

Canonical quantization: expand Φ in a complete set of normalizable modes, satisfying appropriate boundary conditions

Inner Product:
(Klein-Gordon)

$$(\Phi_1, \Phi_2) = -i \int_{\Sigma_t} dz\, d\vec{x} \sqrt{g}\, g^{tt} (\Phi_1^* \partial_t \Phi_2 - \Phi_2 \partial_t \Phi_1^*)$$

Σ_t const. time slice

Can check: (Φ_1, Φ_2) independent of t.
(this was done in problem set 7, problem 1)

$z^{\Delta} \to 0$ as $z \to 0$ is always normalizable

$\Delta = \frac{d}{2} + \nu > 0$

$z^{d-\Delta}$ is non-normalizable for $\nu \geq 1$
is normalizable for $0 \leq \nu < 1$

Boundary conditions:

$\nu \geq 1$: $A = 0$

$0 \leq \nu < 1$: $A = 0$ "standard quantization"
or $B = 0$ "alternative quantization"
(or mixed)

"normalizable" behavior specified by quantization.

(3) Normalizable modes: Used to build up Hilbert space in the bulk

$\Updownarrow$

States of the boundary theory

4) Non-normalizable modes are <u>not</u> part of the Hilbert Space. If present, they should be considered/viewed as defining the background.

In standard quantization: $A \neq 0$

$\rightsquigarrow$ A is boundary "value" of the field

If $A(x) = \varphi(x) \Rightarrow S_{\text{boundary}}$ should contain a term: $\int d^d x\, \varphi(x)\, \mathcal{O}(x)$

$\Rightarrow$ non-normalizable modes determine the boundary theory itself

i.e. two solutions with the same non-normalizable modes describe <u>different states</u> of the same theory.

but two solutions with different non-normalizable modes describe <u>different theories</u>

$$\int d^d x\, \varphi(x)\mathcal{O}(x) \longleftrightarrow \varphi(x) = \lim_{z\to 0} z^{\Delta - d}\, \Phi(z,x) \quad (*)$$

(5) Relation $(*)$ implies that Δ is the scaling dimension of $\mathcal{O}$

$$x \to x'^{\mu} = \lambda x^{\mu}$$

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^{-\Delta} \mathcal{O}(x)$$

Δ: scaling dimension of $\mathcal{O}$

boundary scaling: $\longleftrightarrow$ bulk isometry:

$$x^{\mu} \to x^{\mu'} = \lambda x^{\mu} \qquad\qquad x^{\mu} \to x^{\mu'} = \lambda x^{\mu}$$

$$z \to z' = \lambda z$$

$$\rightsquigarrow \quad \begin{array}{ccc} \Phi(z, x) & \leftrightarrow & \mathcal{O}(x) \\ \downarrow & & \downarrow \\ \Phi'(z', x') & \longleftrightarrow & \mathcal{O}'(x') \end{array}$$

$$\int d^d x' \, \varphi'(x') \, \mathcal{O}'(x') = \int d^d x \, \varphi(x) \, \mathcal{O}(x)$$

Φ : scalar

$$\Phi'(z', x') = \Phi(z, x)$$

$$\varphi'(x') = \lim_{z' \to 0} (z')^{\Delta - d} \, \Phi'(z', x')$$

$$= \lambda^{\Delta - d} \varphi(x)$$

Further, $d^d x' = \lambda^d d^d x$

$\Rightarrow$ $\underline{\mathcal{O}'(x') = \lambda^{-\Delta} \mathcal{O}(x)}$

For a scalar (standard quantization)

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}$$

(i) $m = 0 \Leftrightarrow \Delta = d$ marginal operator

(ii) $m^2 < 0 \Leftrightarrow \Delta < d$ relevant operator

(iii) $m^2 > 0 \Leftrightarrow \Delta > d$ irrelevant operator

$$\int d^d x \, \phi(x) \mathcal{O}(x) \longleftrightarrow z^{d-\Delta} \phi(z) + \cdots$$

$$UV \longleftrightarrow z \to 0$$

(i) does not change $\quad$ const.

(ii) less and less important $\quad \to 0$

(iii) more and more important $\quad \to \infty$

Before:

$\mathcal{O}(x) \longleftrightarrow \Phi(z,x)$

scalar field of m^2

$z \to 0, \quad \Phi(z,x) = A(x)\, z^{d-\Delta} + B(x)\, z^{\Delta}$

$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2 R^2}$

normalizable modes $\longleftrightarrow$ states

non-normalizable modes $\longleftrightarrow$ action (theory)

standard: (A term non-normalizable)

$A(x) \longleftrightarrow \int \varphi(x)\, \mathcal{O}(x) \qquad (\varphi = A)$

$B(x) \longleftrightarrow \langle \mathcal{O} \rangle$

$\delta = \Delta$

alternative quant: $(0 \leq \nu < 1)$

choose $B(x)$ term to be non-normalizable

$B(x) \longleftrightarrow \int \varphi(x)\, \mathcal{O} \qquad \varphi = B$

$A(x) \longleftrightarrow \langle \mathcal{O} \rangle, \quad \delta = d - \Delta$

different CFTs $\{\mathcal{O}\}$

$\Updownarrow$

different gravities with $\{\varphi\}$

<u>conserved currents:</u>

1) $\underline{J^\mu}$ (global $U(1)$ internal symmetry) $\leftrightarrow A_\mu$ gauge field

$\longrightarrow Q = \int_\Sigma d^{d-1}x\, J^0 \Rightarrow [J] = d-1$

2) $\underline{T^{\mu\nu}} \longrightarrow E = \int d^{d-1} T^{00} \rightsquigarrow [E] = 1 \Rightarrow [T] = d$

$\updownarrow$

$h_{\mu\nu}$ (metric perturbations)

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(1) Suppose we deform the CFT

by $\int a_\mu(x) J^\mu(x)\, d^dx \quad (*)$

$a_\mu \equiv A_\mu \big|_{\partial(AdS)}$

since J^μ is conserved, $(*)$ is invariant

under $a_\mu \to a_\mu(x) + \partial_\mu \Lambda(x)$

$\rightsquigarrow A_\mu$ should have some gauge transformation

$\Rightarrow$ Maxwell field

Conversely: Start on gravity side with:

$$-\frac{1}{4}\int d^{d+1}x \sqrt{-g}\, F_{MN} F^{MN} + \dots \qquad A_M = (A_z, A_\mu)$$

$\Rightarrow \delta = d-1$

$z \to 0 \quad A_\mu = a_\mu + b_\mu z^{d-2}$

(2) Add $\int h_{\mu\nu} T^{\mu\nu} d^d x$ to the boundary action

$\Leftrightarrow$ deforming boundary metric $\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$

$$ds^2 = \frac{R^2}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right)$$

$$\xrightarrow[z \to 0]{} \frac{R^2}{z^2} \underbrace{\eta_{\mu\nu} dx^\mu dx^\nu}_{\text{boundary metric}} \quad \cancel{\Rightarrow ds^2 = R^2}$$

so now:

$$ds^2 = \frac{R^2}{z^2}\left((g_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu + dz^2\right)$$

$z \to 0$

$\Rightarrow T_{\mu\nu}$ should correspond to bulk metric perturbations

Conversely:

Using linearized Einstein equations and finding boundary behaviour:

$$g_{MN} = g^{AdS}_{MN} + h_{MN} \quad \text{and finding } h_{MN} \text{ near } z \to 0$$

$$h_{\mu\nu} = \frac{a_{\mu\nu}}{z^2} + b_{\mu\nu}\, z^{d-2} \quad \text{as } z \to 0$$

scaling argument gives $\delta = d$

More generally:

Any bulk field Φ with n indices

$$\Phi(x,z) = A(x)\, z^{d-\Delta-n} + B(x)\, z^{\Delta-n}$$

$$\text{boundary source} = \lim_{z \to 0} z^a\, \Phi(z,x)$$

$$a = \Delta + n - d$$

Δ = dim of corresp. bulk operator

3.1.3 Euclidean Correlation Functions

Basic observables of a CFT:
correlation functions of local ops.

large-N: single-trace ops.

Recall: $\langle \mathcal{O}\mathcal{O}\rangle_c \sim O(1) + \cdots$

$\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle_c \sim O(1/N) + \cdots$

$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle \sim O(N^{2-n}) + \cdots$

leading behavior suggests a <u>tree theory</u> with coupling $1/N$.

convenient to consider Euclidean correlation functions

$t = -i\tau$

can consider generating functional:

$$Z_{CFT}[\varphi] \equiv \underbrace{\begin{pmatrix}\text{order} \\ \text{not mattering}\end{pmatrix}}\ \left\langle e^{\int d^d x\, \varphi(x)\, \mathcal{O}(x)} \right\rangle_c \quad (*)$$

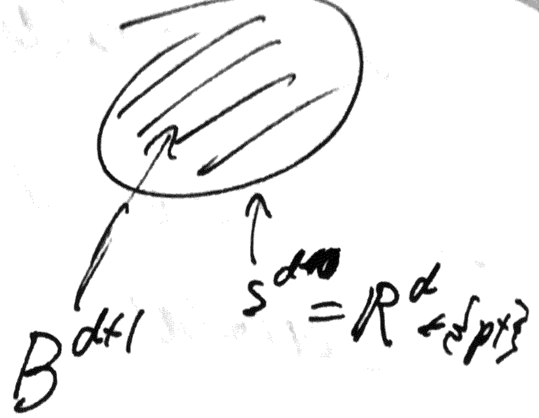
$\mathcal{O}$: collections of all single-trace ops

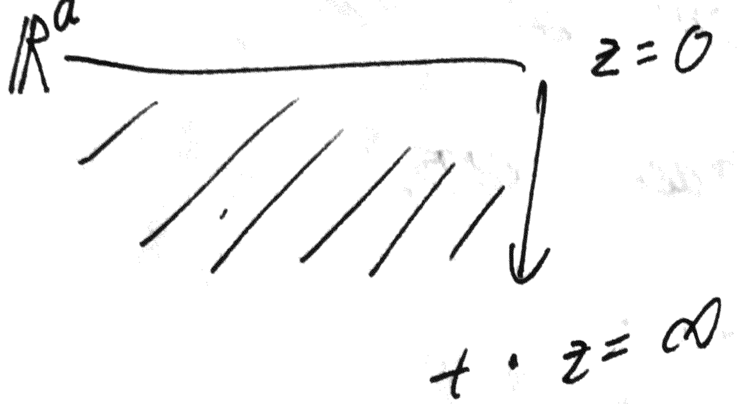
φ: sources

$\rightsquigarrow$ Analytically continue AdS to $\underbrace{\text{Euclidean}}_{\text{space}}$

$$ds^2 = \frac{R^2}{z^2}\left(d\tau^2 + d\vec{x}^2 + dz^2\right) \leftarrow \text{covers } \underline{\text{full}} \text{ Euclidean } \cancel{\text{AdS}}$$

$$ds^2 = \frac{R^2}{z^2}\left(d\tau^2 + d\vec{x}^2 + dz^2\right) \to$$





Given that $\mathcal{O} \leftrightarrow \Phi$

$\varphi(x) \leftrightarrow \Phi\big|_{\partial AdS}$

$$\boxed{Z_{CFT}[\varphi] = Z_{bulk}\left[\Phi\big|_{\partial(AdS)} = \varphi\right]}$$

in the sense of $\boxed{\lim_{z\to 0} z^{\alpha}\, \Phi(z,x)}$

We know how to define this

not known how to define in general

In the low energy limit, $G_N \to 0$, $\alpha' \to 0$

$$Z_{bulk} = \int D\Phi\, e^{S_E[\Phi]}$$

and we can evaluate this perturbatively around pure (Euclidean) AdS

General n-point function:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{\delta^n \log Z_{bulk}}{\delta\varphi_1\cdots\delta\varphi_n(x_n)}\Big|_{\varphi=0}$$

Recall around AdS:

$$S_{bulk} = \int d^{d+1}x\,\sqrt{g}\left[-\frac{1}{2}(\partial\Phi)^2-\frac{1}{2}m^2\Phi^2-\kappa\Phi^3-\kappa^2\Phi^4+\cdots\right]$$

$$(\kappa\sim\sqrt{G_N}\sim 1/N)$$

let Φ_i collectively denote all perturbations around AdS, including metric and all matter fields

 $\sim 1/N$  $\sim\kappa^2\sim 1/N^2$

 $\sim 1/N^3$

$$\partial^2\Phi_1-m^2\Phi_1=\kappa\Phi_0^2,\quad \Phi_0=\int d^dx'\,K(z,x;x')\varphi(x')$$

$$\Phi_1=\int dx'dz'\,G(x,z;x',z')\,\kappa\,\Phi_0^2(x',z')+\cdots$$

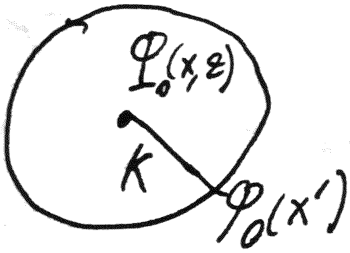
From this structure, we get that the free-level is

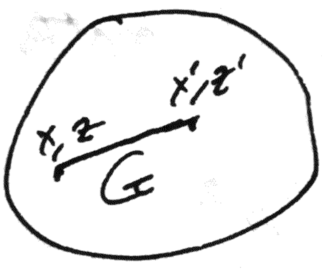
$$\log Z_{tree}[\varphi] = \varphi^2 + x\varphi^3 + x^2\varphi^4 + \dots$$

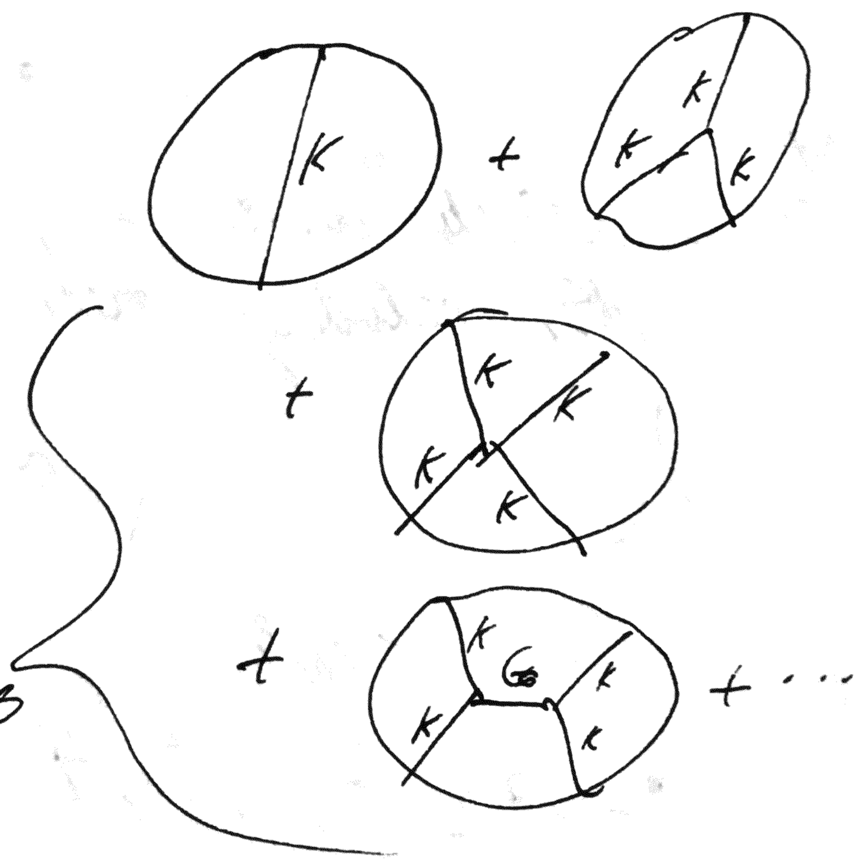
K : boundary to bulk propagator

G : bulk to bulk propagator

↑ demands fields fall off at ∞







Witten Diagrams

Dec 3, 2018

$$\left\langle \exp\left[\int O(x)\varphi(x)\right]\right\rangle = Z_{bulk}\left[\Phi\big|_{\partial AdS} = \varphi(x)\right] \quad (*)$$

$$Z = \int D\Phi \, \exp\left[S_E[\Phi]\right]$$

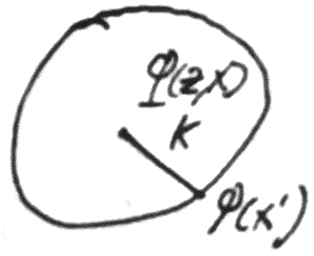
$$S_E[\Phi] = -\int d^{d+1}x \sqrt{g}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \kappa\Phi^3 + \kappa^2\Phi^4 + \dots\right]$$

$$\kappa \sim G_N^{1/2} \sim 1/N$$

$$\log Z_{bulk}[\Phi] = \log Z_{tree}[\Phi] + \log Z_{1\text{-loop}}[\Phi] + \dots$$

$$\Phi_0(z,x) = \int d^d x' \, K(z,x;x')\varphi(x')$$

$$\lim_{z\to 0} \Phi_0(z,x) = z^{d-\Delta}\varphi(x)$$



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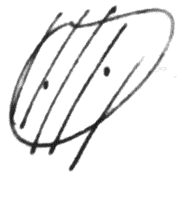
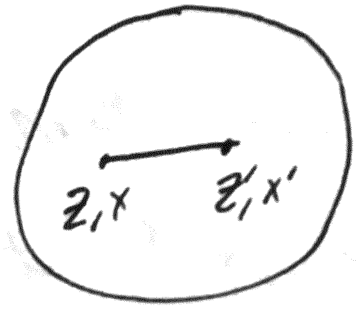
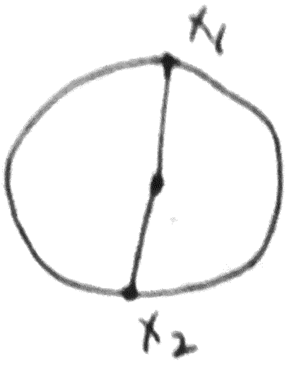
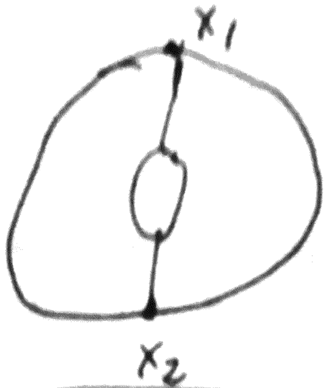


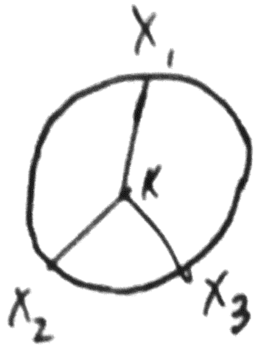


$$(\partial^2 - m^2)G(z,x;z',x') = \frac{1}{\sqrt{g}}\delta(z-z')\delta^{(d)}(x-x')$$

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle = \frac{\delta \log Z_{bulk}}{\delta \varphi(x_1)\, \delta \varphi(x_2)} \Bigg|_{\varphi = 0}$$







$+ \cdots$

Remarks:

(1) Both sides of $(*)$ are divergent, even at tree-level, in the bulk

LHS: Usual UV divergences of a QFT $\leftarrow$ IR/UV

RHS: Volume divergences + asymptotic behavior of φ $\to$ divergences as $z \to 0$

$\rightsquigarrow$ We need to renormalize them:

$$\log Z^{(R)}_{CFT} = \log \underbrace{Z_{CFT}}_{\text{bare}} + S_{ct}[\underset{\text{local}}{\varphi}]$$

$$\log Z^{(R)}_{bulk} = \underbrace{\log Z_{bulk}}_{\text{bare}} + \underset{\text{local}}{S_{ct}(}$$

(2) One-point function

consider CM of 1 dof.

$$S[x_c] = \int_{t_0}^{t_1} dt\, L[x(t), \dot{x}(t)]$$

$$x_c(t_0) = X_0$$
$$x_c(t_1) = X_1$$

$$\delta S = p_1 \delta X_1 - p_0 \delta X_1 \Rightarrow \frac{\delta S}{\delta X_1} = p_1$$

$$\langle O(x) \rangle = \frac{\delta \log Z_{tree}}{\delta \varphi(x)} \qquad \log Z_{tree} = S_E[\Phi_c]$$

$$\Phi_c \Big|_{\partial AdS} = \varphi(x)$$

$$\langle O(x) \rangle = \frac{\delta S_E^{(R)}[\Phi_c]}{\delta \varphi(x)} = \lim_{z \to 0} z^{\Delta - d} \frac{\delta S_E^{(R)}(\Phi_c)}{\delta \Phi_c(z, x)}$$

$$= \lim_{z \to 0} z^{\Delta - d}\, \Pi^{(R)}(\Phi_c)$$

Π: canonical momentum for Φ treating z as "time"

- boundary-to-bulk prop.

$$K(z,x;x') : (\partial^2 - m^2)\, K(z,x;x') = 0$$

$$K(z\to 0, x; x') = z^{d-\Delta}\, \delta^{(d)}(x-x')$$

- bulk-to-bulk prop.

counterpart of standard Flat space prop.

$G(z,x;z',x')$ is normalizable if either $z, z' \to 0$

$$G(z,x;z',x') \propto z^{\Delta} \qquad z\to 0$$

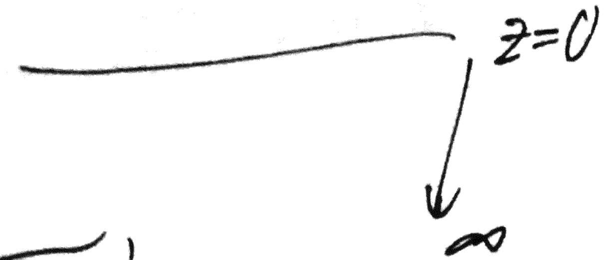
Both G and K are found by going into momentum space

e.g. $$K(z,x;x') = \int \frac{d^d k}{(2\pi)^d}\, K(z,k)\, e^{ik(x-x')}$$

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z K\right) - \left(k^2 z^2 + m^2 R^2\right) K = 0$$

with $K(z,k) = z^{d-\Delta} + \dots$ as $z\to 0$

Find $K(z,x;x')$ directly in coordinate space



$$\frac{R^2}{z^2}\left(d\tau^2 + d\vec{x}^{\,2} + dz^2\right)$$

$$P = (z=\infty)$$

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$$\hat{K}(z) \equiv K(z,x;P)$$

$$\left(z^2\partial_z^2 + (1-d)\, z\partial_z + m^2R^2\right)\hat{K} = 0$$

$$\underline{\hat{K} = a z^{d-\Delta} + b z^{\Delta}} \Rightarrow \hat{K} = b z^{\Delta}$$

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2}$$

<u>Inversion</u>

$$z \to \frac{z}{z^2+x^2}$$

$$x^\mu \to \frac{x^\mu}{z^2+x^2}$$

$$I: P \to \begin{matrix} z'=0 \\ x'=0 \end{matrix}, \quad K(z,x;x') = b\left(\frac{z}{z^2+(x-x')^2}\right)^{\Delta}$$

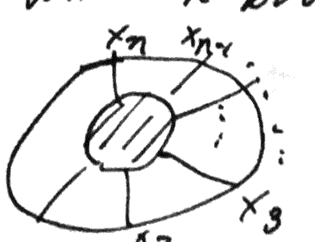
$$\Rightarrow b = \frac{\Gamma(\Delta)}{\Gamma(\nu)}\pi^{-d/2}$$

<u>Can also show</u> (without solving G)

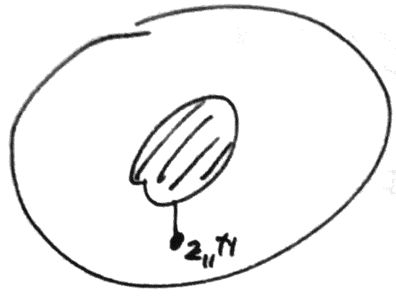
$$K(z,x;x') = \lim_{z'\to 0} 2\nu z'^{-\Delta} G(z,x;z',x')$$

$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle$ = sum over all Feynman diagrams with n boundary endpoints

=



$$\langle \Phi_1(z_1, x_1) \cdots \Phi_n(z_n, x_n) \rangle$$

$=$ 

$$\Rightarrow \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \lim_{\substack{z_1 \to 0 \\ \vdots \\ z_n \to 0}} (2\nu_1 z_1^{-\Delta_1}) \cdots (2\nu_n z_n^{-\Delta_n}) \times \langle \Phi_1(x_1, z_1) \cdots \Phi_n(x_n, z_n) \rangle \qquad (**)$$

Since Lorentzian corr. func. can be obtained from Euclidean ones using the same analytic continuation procedure, $(**)$ must also apply to Lorentz corr.

Recall that: $|0\rangle_{AdS} \leftrightarrow |0\rangle_{CFT}$

$$\{ \Phi(z, x) \cdots |0\rangle_{AdS} \} \Leftrightarrow \{ \mathcal{O}(x) \cdots |0\rangle_{CFT} \}$$

$\rightsquigarrow$ we can identify

$$\boxed{\mathcal{O}(x) = 2\nu \lim_{z \to 0} z^{-\Delta} \Phi(x, z)}$$

$$\langle \Psi | \mathcal{O}(x) | \Psi \rangle = 2\nu \lim_{z \to 0} z^{-\Delta} \underbrace{\langle \Psi | \Phi(x,z) | \Psi \rangle}_{= B(x) z^{\Delta} + \cdots}$$

$$= 2\nu B(x)$$

$$\langle 0 | \mathcal{O}(x_1) \mathcal{O}(x_2) | 0 \rangle = \lim_{\substack{z_1 \to 0 \\ z_2 \to 0}} (2\nu z_1^{-\Delta})(2\nu z_2^{-\Delta}) G_\Sigma(z_1, x_1; z_2, x_2)$$

$$= \lim_{z_1 \to 0} 2\nu z_1^{-\Delta} K(z_1, x_1; x_2)$$

$$= \frac{2\nu b}{|x_1 - x_2|^{2\Delta}}$$

3.1.4 <u>Wilson loops</u>

non-local operators

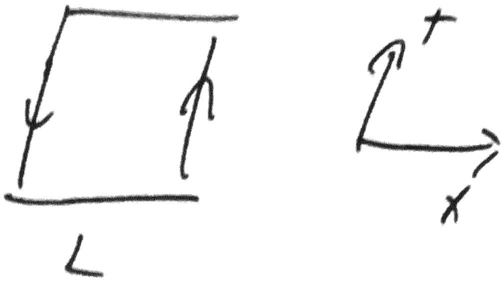
recall:

$$W[C] = \mathrm{Tr}_\rho \, P e^{i \oint_C A_\mu dx^\mu}$$

$\rightsquigarrow$ phase factor associated with transporting an "external" particle in a given rep'n ρ along C

e.g. $\langle 0 | W_\rho(C) | 0 \rangle$, $\langle 0 | W_\rho(C_1) W(C_2) \cdots | 0 \rangle$

Often-used C:



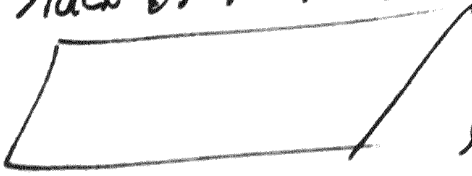
$T \gg L$, expect $\langle W(C) \rangle \propto e^{-iET}$

E = energy of quark-antiquark system at $M \to \infty$

so they don't move

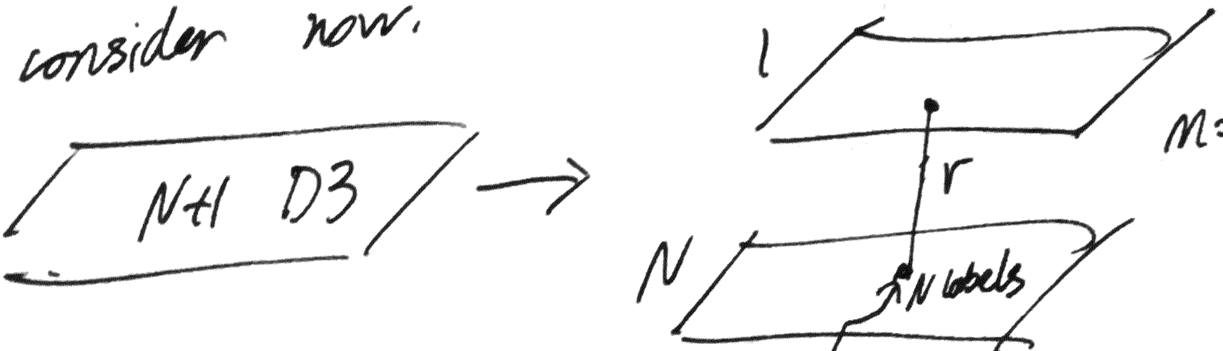
First, we need to understand how to introduce "external quarks" into $\mathcal{N}=4$ SYM

stack of N D3 branes



$\xrightarrow{\text{low-energy limit}}$ $AdS_5 \times S^5$

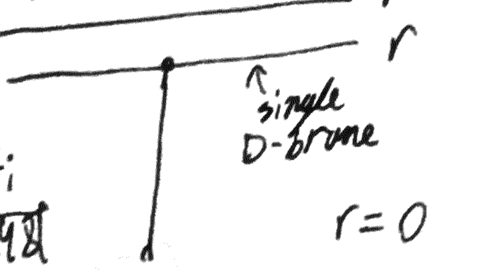
consider now:



$M = \frac{r}{2\pi\alpha'}$

like a "quark" in fundamental rep'n

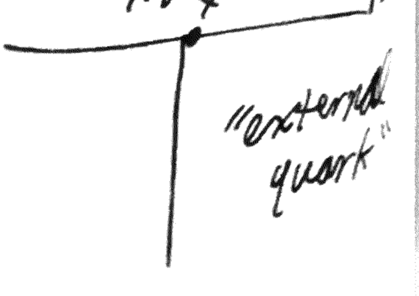
Now take low-energy limit, $r \to 0$, $\alpha' \to 0$ with r/α' fixed



$r = R^2/z$

$\boxed{M = r/2\pi\alpha'}$

taking $r \to 0$, we get



we get:

Dec 5th, 2018 (Notes from Sam Leutheusser)

Parallel transport of such an "external quark" along a closed curve C gives a Wilson loop.

$$W(C) = \mathrm{Tr}\, P \exp\left[i \oint_C A^a_\mu dx^\mu_a\right]$$

$$\tilde{W}(C) = \mathrm{Tr}\, P \exp\left[i \oint_C \left(A_\mu \frac{dx^\mu}{ds} + \vec{n} \cdot \vec{\Phi} \sqrt{\dot{x}^2}\right) ds\right]$$

$\vec{\Phi}$ = six scalars in $\mathcal{N}=4$ SYM

$\vec{n}$ a unit vector on S^5

Since:

1) "quark" is endpoint of a string in AdS, C must be the boundary of a string worldsheet, Σ i.e. $C = \partial\Sigma$

2) $\langle W(C) \rangle$ is the "partition function of this quark"

Guess: $\langle W(C) \rangle = Z_{string}(\partial\Sigma = C)$ $\leftarrow$ single string partition function

$$Z_{\text{string}} = \int_{\partial\Sigma = C} DX^{\mu}(\sigma^{\alpha})\, e^{iS_{\text{string}}},$$

$$S_{\text{string}} \propto \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h}$$

where $h = \det(h_{\alpha\beta})$, $h_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$

$\cancel{h_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \ \alpha,\beta \in \{0,1\}}$ $\alpha \in \{0,1\}$

$g_s \to 0$, $\alpha' \to 0$

neglect other topologies beyond genus 0 (large N)

saddle point approx. (large λ)

$$\Rightarrow Z_{\text{string}} = e^{iS_{\text{string}}[X_{\text{classical}}]}$$

$X_{\text{classical}}$ - solution to worldsheet EOM

$$\Rightarrow \langle W(C)\rangle = e^{iS[X_c]}\big|_{\partial\Sigma = C}$$

action evaluated at classical string solution

Examples|

(1) A static quark $\quad C = {}^{T}\!\uparrow$ in field theory

length T

Field theory: $\langle W(C)\rangle = e^{-iMT}$, M is quark mass

Gravity: $$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \qquad z = \frac{R^2}{r}$$

$$= \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2}dr^2$$

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$\sigma^\alpha = (\sigma, \tau)$, $\sigma = r$ $\tau = t$ ← worldsheet coordinate choice

X^i = const. (trivial solution)

$$ds^2_{ws} = h_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta = -\frac{r^2}{R^2} d\tau^2 + \frac{r^2}{R^2} d\sigma^2$$

← D-brane at r_0

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-h} = -\frac{1}{2\pi\alpha'}\int dt \int_0^{r_0} d\sigma = -\frac{1}{2\pi\alpha'} T r_0$$

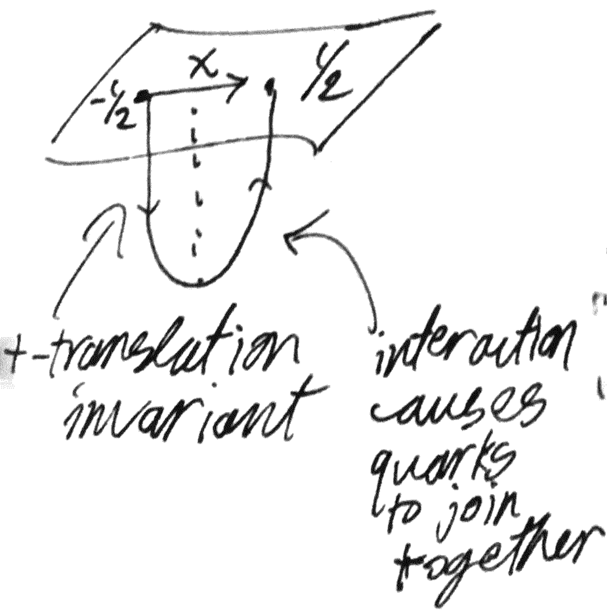
so $S_{NG} = -MT$, $M = \frac{r_0}{2\pi\alpha'}$

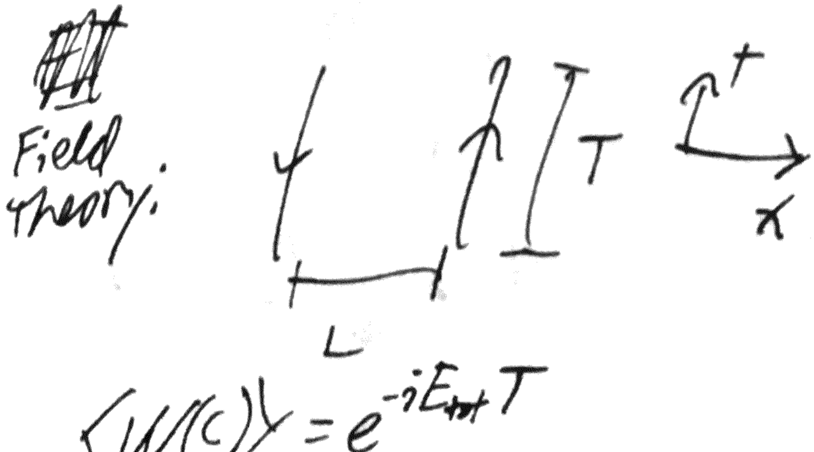
External quark: $r_0 \to \infty \Rightarrow M \to \infty$

Take $r_0 = \Lambda$, $M = \frac{\Lambda}{2\pi\alpha'}$ Λ: UV energy cutoff

$$\Lambda = \frac{R^2}{\varepsilon},\ z = \varepsilon \Rightarrow M = \frac{R^2}{2\pi\alpha'}\frac{1}{\varepsilon} = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{\varepsilon}$$

2) Static Potential between quark/antiquark





$$\langle W(C)\rangle = e^{-iE_{tot}T}$$

$$E_{tot} = 2M + V(L)$$

Choose: $\tau = t$, $\sigma = x$

$X^i = \text{const}$ for $i \neq x$, $Z = Z(\sigma) = Z(X)$

with boundary conditions

$Z(\pm L/2) = 0$

$$ds^2_{ws} = \frac{R^2}{z^2}\Big(-d\tau^2 + \underbrace{(1+(z')^2)}_{\text{"}dx^2\text{"}\ \ \text{"}dz^2\text{"}} d\sigma^2\Big)$$

$$S_{NG} = \frac{2R^2}{2\pi\alpha'} T \int_{\varepsilon}^{L/2} \underbrace{\frac{d\sigma}{z^2}\sqrt{1+(z')^2}}_{\mathcal{L}\text{ indep. of }\sigma}$$

$x \leftrightarrow -x$ sym

$z(\sigma) = z(-\sigma)$

ε as UV cutoff

$$V(L) = \frac{\sqrt{\lambda}}{\pi}\int_{\varepsilon}^{L/2} d\sigma \sqrt{1+(z')^2} - \frac{2\sqrt{\lambda}}{2\pi}\frac{1}{\varepsilon}$$

$\mathcal{L}$ is extremized by $z'\Pi_z - \mathcal{L} = \text{const}$, $\Pi_z = \frac{\partial \mathcal{L}}{\partial z'}$

at $\sigma = 0$, $z(0) = z_0$

$$\Rightarrow (z')^2 = \frac{z_0^4 - z^4}{z^4}, \quad z_0 = \frac{L\sqrt{\pi}}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$\rightsquigarrow V(L) = \frac{\sqrt{\lambda}}{\pi}\left[z_0^2 \int_{\varepsilon}^{z_0} \frac{dz}{z^2}\frac{1}{\sqrt{z_0^4 - z^4}} - \underbrace{\frac{1}{\varepsilon}}_{\text{cancels UV div}}\right] \Rightarrow V(L) = -\frac{\sqrt{\lambda}}{L}\left(\frac{4\pi^2}{\Gamma^4(1/4)}\right)$$

Remarks:

(a) $V(L)$ is finite, negative $\rightsquigarrow$ attraction of quark-antiquark

(b) $V(L) \propto 1/L$ (scale invariance, only energy is $1/L$)

(c) $V(L) \propto \sqrt{\lambda}$ (strong coupling result, at weak coupling $V(L) \propto g^2_{YM} N$)

(d) $z_0 \propto L \Rightarrow$ deeper in bulk $\leftrightarrow$ larger L (IR/UV connection)

Dec 10th, 2018 (Notes from Sam Leutheusser)

3.2 Finite Temperature

CFT		Gravity
states	$\longleftrightarrow$	states = 
$\vert 0\rangle$	$\longleftrightarrow$	$\vert 0\rangle$ in pure AdS

Fock space around geometry

$E - E_g \approx O(G_N^0)$

Finite Temperature:

SU(N) gauge theory in flat space

$\rightarrow \varepsilon \propto N^2,\ s \propto N^2, \ldots$ $\Rightarrow \varepsilon \sim 1/G_N$

Backreaction: $G_N \varepsilon \sim O(1)$

gravity backreaction so this is a new geometry!

Q: What does the thermal state in CFT correspond to?

criteria it should satisfy:

(1) asymptotic AdS (normalizable, since finite T gives same theory, different state)

(2) satisfy all laws of thermodynamics

(3) translationally and rotationally invariant along boundary directions

Candidates:

1. Thermal AdS: $ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$

$-dt^2 \leftrightarrow d\tau^2$, with $\tau \sim \tau + \beta$

geometry is singular as $z \to \infty$

circle of τ goes to zero size $\Rightarrow$ singularities

2. A Black Hole with event horizon that is topologically R^{d-1}

Ansatz: $ds^2 = \frac{R^2}{z^2}(-f(z)dt^2 + d\vec{x}^2 + g(z)dz^2)$

flat horizon

Flat horizon due to Poincare symmetry

$\rightsquigarrow$ Einstein equations: $f = 1/g = 1 - \frac{z^d}{z_0^d}$ (in AdS_{d+1})
$(\Lambda < 0)$

$z_0 = \text{const} \rightsquigarrow$ horizon at $z = z_0$

standard trick of going to Euclidean time

$\rightsquigarrow$ require Euclidean sol'n smooth

$\Rightarrow \beta = \frac{1}{T} = \frac{4\pi}{d} z_0 \Rightarrow T = \frac{d}{4\pi}\frac{1}{z_0}$ (measured in units of t)

$z = 0$

$1/T \sim z = z_0$ - - - - - - - horizon

Higher T horizon $\Rightarrow z = 0$ probes high energy

Lower T horizon $\Rightarrow z = \infty$ probes low energy

(IR/UV connection)

Thermodynamics

$$S_{BH} = \frac{A_{hor}}{4G_N}$$

For AdS_5, $d=4$

entropy density

$$A_{hor} = \frac{R^3}{z_0^3}\underbrace{\int d\vec{x}}_{V = \text{spatial volume of boundary}} \Rightarrow s = \frac{R^3}{4 z_0^3 G_N} = \frac{\pi^2}{2} N^2 T^3 \quad (*)$$

5d G_N with $G_N / R^3 = \pi / 2N^2$

$(*)$ is <u>also</u> a prediction of entropy density on $\mathcal{N}=4$ SYM in $N \to \infty$, $\lambda \to \infty$ limit.

Read $\langle T_{\mu\nu}\rangle_\beta$ from metric $\Rightarrow \langle T_{\mu\nu}\rangle_\beta \propto 1/z_0^d \propto T^d$

matches CFT prediction

Can also use thermodynamic relations:

$$s = -\frac{\partial F}{\partial T} \Rightarrow F = -\frac{\pi^2}{8} N^2 T^4 \Rightarrow E = F + Ts = \frac{3\pi^2}{8} N^2 T^4$$

Compare with free theory

$$s_{\lambda=0} = (\underset{\substack{\uparrow \\ \text{2 on-shell } A_\mu \\ +6 \text{ scalars}}}{8} + \underset{\substack{\uparrow \\ \text{8 fermions}}}{8} \cdot \underset{\substack{\uparrow \\ \text{fermion contrib.}}}{\tfrac{7}{8}})\, \frac{2\pi^2}{45} T^3 N^2 = \frac{2}{3}\pi^2 N^2 T^3$$

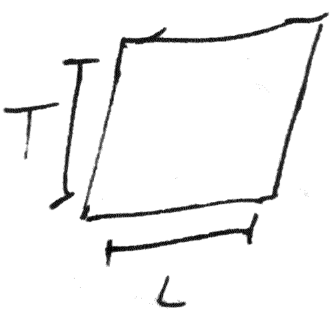
$$\Rightarrow \boxed{\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{3}{4}}$$

Many examples of CFT duals are known for $d=4$. In all known cases we get

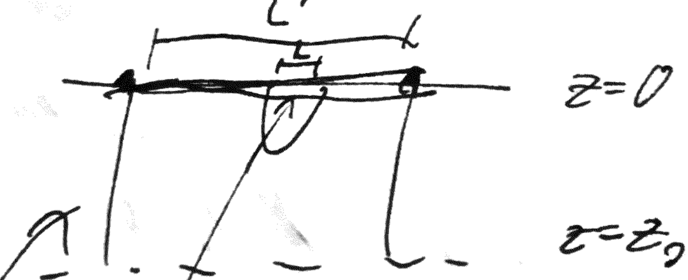
$$\frac{S_{strong}}{S_{free}} = \frac{3}{4}h, \text{ with } 8/9 < h < 1.09$$

$\rightsquigarrow$ ratio is always $\approx 3/4$ for CFT duals (in ∞ of theories) $\leftarrow$ no idea why

$$\langle W(C)\rangle_\beta \sim e^{+iET}, \quad E = 2M + V(L)$$



$\rightsquigarrow$



small L, short distance physics doesn't feel temp. in CFT (doesn't see z_0 in the bulk)

at large L', minimal surface ends on horizon $\Rightarrow$ screened quarks, "finite T plasma screens"

So far, CFT is on $\mathbb{R}^d$, dual to black brane

~~what happ~~ $\rightsquigarrow$ scale inv. theory, T is only scale, so all T are the same, related by scaling so T sets units

ISG

What happens if we consider global AdS?
$\Rightarrow$ CFT is on $\mathbb{R}\times S^{d-1}$ at finite T

Take the sphere radius to be R (just sets scale)
At finite T, there is dimensionless param. RT

<u>Some important features:</u>

(1) Thermal AdS is now allowed

global AdS: $ds^2 = -\left(1+\frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega^2_{d-1}$

$r\to\infty$: $ds^2 = \frac{r^2}{R^2}\underbrace{\left(-dt^2 + R^2 d\Omega^2_{d-1}\right)}_{\text{boundary metric}}$

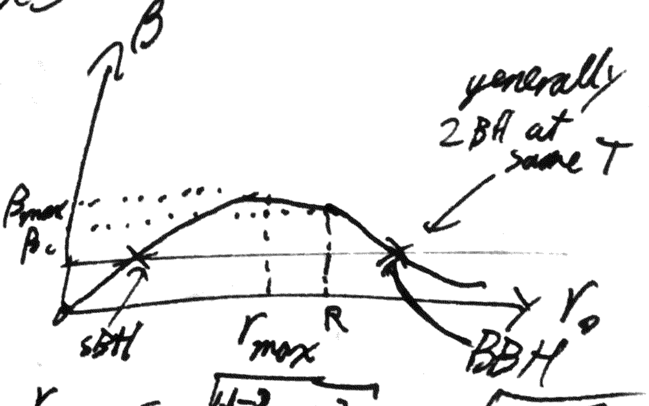
$t\to -i\tau$, $\tau \sim \tau+\beta$ (no singularity at $r\to 0$ since g_{tt} finite)

(2) $ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2_{d-1}$

$f = 1 - \underbrace{\frac{\mu}{r^{d-2}}}_{\substack{\text{Schwarzschild}\\ \text{BH of mass } M}} + \frac{r^2}{R^2}$ ← for asymptotically AdS

horizon at $r=r_0$, $f(r_0)=0$

$\beta = \frac{4\pi}{f'(r_0)} = \frac{4\pi r_0 R}{d\cdot r_0^2 + (d-2)R^2}$ →



$\Rightarrow r_{max} = \sqrt{\frac{d-2}{d}R^2}$

(i) $\beta_{max} \Rightarrow T_{min}$

(ii) two BH solutions at given β

(3) We find

(i) For $T < T_{min} \rightarrow$ no BH, only thermal AdS (TAdS)

(ii) For $T > T_{min} \rightarrow$ three possibilities:

TAdS, SBH, BBH

$$e^{-\beta F_{CFT}} = Z_{CFT}(\beta) = Z_{grav}(\beta) = \int D\Phi \, e^{S_E[\Phi]}$$

$$= \sum_{\text{saddles}} e^{S_E[\Phi_c]}$$

Dominant saddle has largest $S_E[\Phi_c]$

this solution dominates

$$S_E = \frac{1}{16\pi G_N} \int \left[(R - 2\Lambda) + \mathcal{L}_{matter}\right] \propto N^2$$

pure AdS: $S_E = 0$ (have to renormalize s.t. this holds)

TAdS : $S_E = 0 \cdot N^2 + O(N^0)$ (b.c. this only differs by global $\tau \sim \tau + \beta$ so curvature terms are locally the same)

BBH:
SBH: $\propto \# N^2$

sign of # determines if these are dominant

$\rightarrow$ complicated calculation

short cut: $\omega_{d-1} = \text{Vol of } S^{d-1}$

$$s = \frac{\omega_{d-1}\, r_0^{d-1}}{4 G_N} \xrightarrow{\text{integrate}} S = -\frac{\partial F}{\partial T} = -\frac{\partial F}{\partial r_0}\frac{\partial r_0}{\partial T}$$

$$\Rightarrow F = \frac{\omega_{d-1}}{16\pi G_N}\left(r_0^{d-2} - \frac{r_0^{d}}{R^2} \right)$$

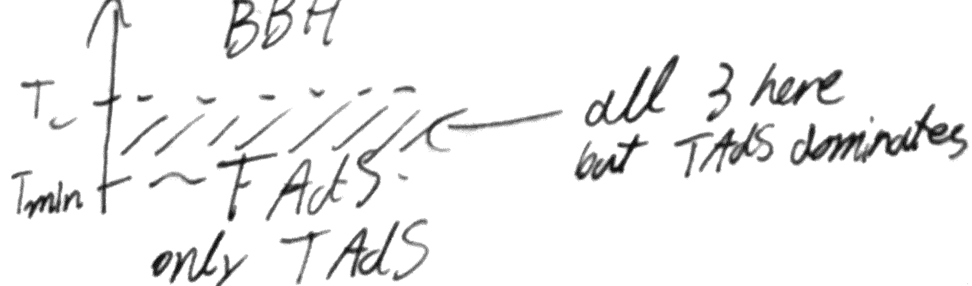
$F_{BH} > 0$ when $r_0 < R$ $\leftarrow \beta_c = \beta(r_0 = R) = 1/T_c$

$F_{BH} > 0$ when $r_0 > R$

$T_c > T_{min}$, for $T < T_c$ thermal AdS dominates

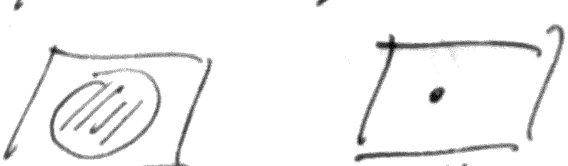
for $T > T_c$ BBH dominates

always $S_E(BBH) > S_E(SBH)$ and $S_E(SBH) < 0$



(5) BBH has positive specific heat

SBH has negative specific heat



BBH → stable

SBH → unstable → evaps (doesn't know its in box)

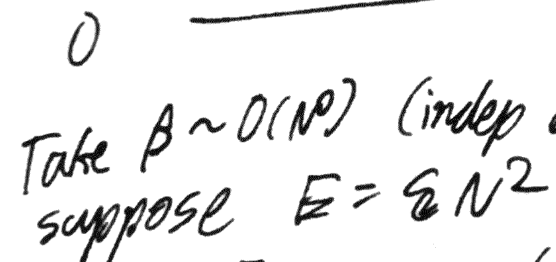
(6) Since physics only depends on RT

keep RT fixed, $T \to \infty$ = $\underbrace{\text{keep } RT \text{ fixed, } R \to \infty}_{\text{flat space limit}}$

(7) Hawking-Page transition
(sometimes called "deconfinement transition")

$T < T_c \Rightarrow F_{CFT} \sim O(N^0)$
$T > T_c \Rightarrow F_{CFT} \sim O(N^2)$
$\}$ 1^{st} order phase transition

(8) Physics underlying HP transition:
A free theory of two matrices A, B $(N \times N)$
$\rightsquigarrow 2N^2$ harmonic oscillators w/ frequency $\omega = 1$



} evenly spaced Energy levels

Density of states:
$D(E) \sim O(N^0)$ when $E \sim N^0$
$D(E) \sim e^{\alpha(N^2)}$ when $E \sim N^2$

$$Z = \int dE \, D(E) e^{-\beta E}$$
$D(E) \approx e^{S(E)}$

Take $\beta \sim O(N^0)$ (indep of N)
suppose $E = \varepsilon N^2 \Rightarrow S(E) = F(\varepsilon) N^2$
$\rightsquigarrow e^{-\beta E} D(E) = e^{(F(\varepsilon) - \beta\varepsilon) N^2}$
$\rightsquigarrow$ For $F(\varepsilon) < \beta\varepsilon$ highly excited ($E \sim N^2$) states won't contribute
$\Rightarrow Z$ receives dominant contributions from $E \sim O(N^0)$
$\Rightarrow F = O(N^0)$

IGO | IF $F(\varepsilon) > \beta\varepsilon$, $F = O(N^2)$

3.3 Holographic Entanglement Entropy

- Entanglement entropy:

divide system $A+B$

Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\Psi\rangle = \sum_n \psi_n(A)\, \chi_n(B)$$

A and B in state $|\Psi\rangle$ are entangled if $|\Psi\rangle$ cannot be written as a simple product of states of A, B

EE: a measure to quantify how much A and B are entangled

$$\rho_A = \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|$$

$$\rightsquigarrow \boxed{S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A} \geq 0$$

$S_A = 0 \Leftrightarrow \rho_A$ is a pure state

$\Leftrightarrow |\Psi\rangle$ can be written as a simple product

For a pure state: $S_A = S_B$ in general

If AB is in a mixed state, we do not in general have $S_A = S_B$

Example: 2 spin system $\uparrow \; \uparrow$ (A, B)

a) $|\psi\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_B + |\downarrow\rangle_B\right)$

$\rightsquigarrow$ not entangled

b) $|\psi\rangle = \cos\theta\, |\uparrow\downarrow\rangle + \sin\theta\, |\downarrow\uparrow\rangle$

$\rho_A = \cos^2\theta\, |\uparrow\rangle\langle\uparrow| + \sin^2\theta\, |\downarrow\rangle\langle\downarrow|$

$\rightsquigarrow S_A = -\cos^2\theta \log\cos^2\theta - \sin^2\theta \log \sin^2\theta$

maximally entangled

Some important properties:

(1) Subadditivity

$$|S(A) - S(B)| \leq S(AB) \leq S(A) + S(B)$$

(2) Strong subadditivity:

$$\cancel{S(AB)} \quad S(AC) + S(BC) \geq S(ABC) + S(C)$$

$$S(AC) + S(BC) \geq S(A) + S(B)$$

Entanglement Entropy in <u>many-body systems</u>:

If $\overset{AB}{H} = \underset{\sim}{H_A} + \underset{\sim}{H_B}$ $\Rightarrow$ ground state is unentangled

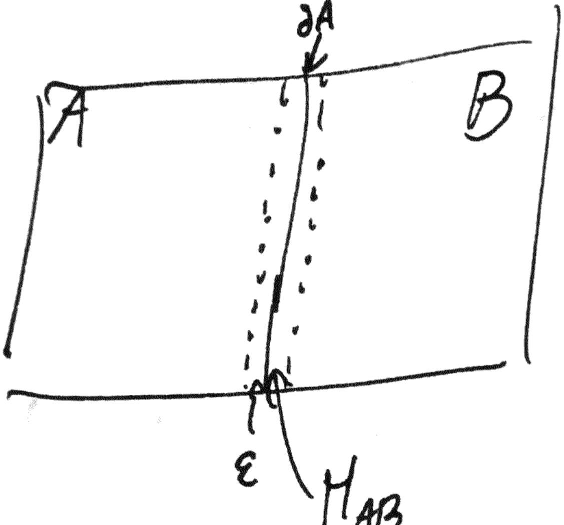
$\rightsquigarrow$ start with unentangled initial state then the system remains unentangled

$\Rightarrow$ interactions are <u>crucial</u> for generating entanglement

Now consider $H = H_A + H_B + \underset{\sim}{H_{AB}}$ $\rightsquigarrow$ ground state is <u>entangled</u>

$\rightsquigarrow$ entanglement will be generated from time evolution

In realistic condensed matter systems and QFTs, H and H_{AB} are <u>local</u>



take ε here to be the lattice spacing

$\rightsquigarrow$ H_{AB} only involves d.o.f. near $\partial A = -\partial B$

e.g. take $H = \sum_{ij} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$

QFT: $\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4$

<u>One Finds</u>: in general in the ground state for a local H,

$$S_A = \# \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}} + \cdots \qquad (*)$$

i.e. entanglement between A and B is dominated by short-range entanglement near ∂A, where $\mathcal{H}_{AB}$ is supported

Area law is universal, $\#$ is <u>not</u> universal, and depends on details of UV theory.

sub-leading terms in (*), which encode long-range entanglement, can provide important characterization of a system

Example:

in (2+1)-dimensions

(1) characterize topological order — realized by X.G. Wen, M. Levin and independently by J. Preskill, A. Kitaev

typical gapped systems:

$\rightsquigarrow$ contains only short-range entanglement

but in topologically ordered systems:

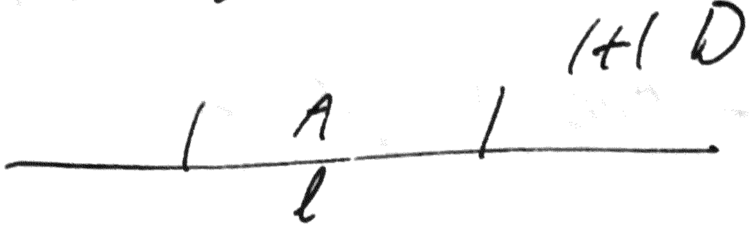
ground state can have subtle long-range correlations

$$\rightsquigarrow S_A = \# \frac{L}{\varepsilon} - \gamma$$

γ: indep. of shape and size of A — topological entanglement entropy

(2) Characterize # d.o.f. of a system of relativistic QFTs

entanglement entropy in CFT $\leadsto$ 1+1 D



$$S_A = \frac{c}{3} \log \frac{\ell}{\varepsilon}$$

where c is the CFTs central charge

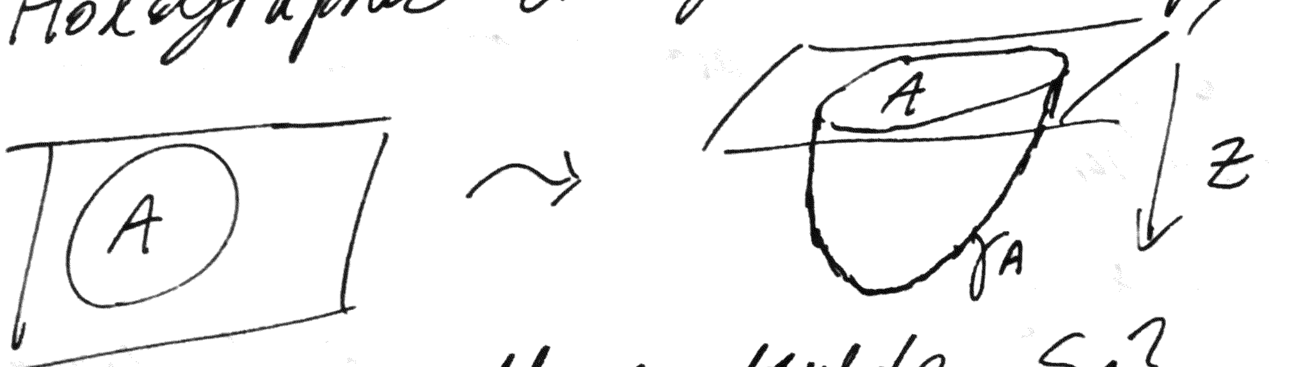
in (2+1)-D CFT:

$$S_A = \# \frac{L\, \partial A}{\varepsilon} - \gamma$$

depends on shape of A

minimized at A = circle

Holographic entanglement entropy



How would we calculate S_A?

Proposal: Find the minimal area surface γ_A which extends into the bulk with ∂A as its boundary

Ryu-Takanagi:

Then $\boxed{S_A = \frac{\text{Area}(\gamma_A)}{4 G_N}}$ (**)

this diverges as $1/\epsilon^{d-2}$, where ϵ is the z-cutoff

∂A: $d-2$ dim

A, γ_A: $d-1$ dim

in AdS_{d+1}

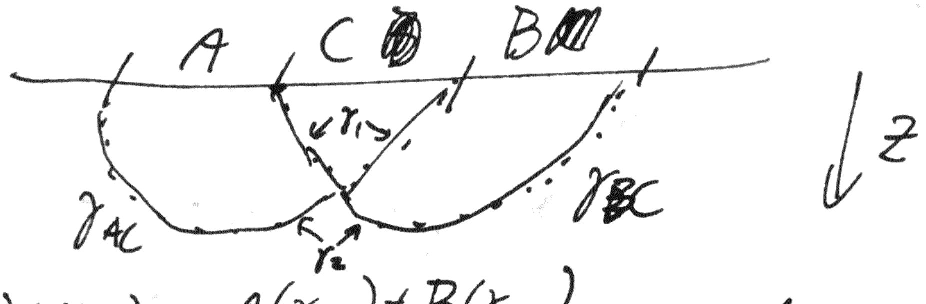
S_A: dimensionless

This formula is <u>very valuable</u>, as entanglement entropy is extremely difficult to calculate even in non-interacting QFTs, but this minimal surface area is relatively easy to calculate.

Things to check:

- strong subadditivity:

$$S(AC) + S(BC) \geq S(ABC) + S(C)$$

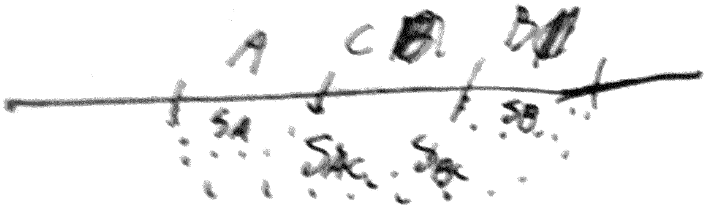


$$S(AC) + S(BC) = A(\gamma_{AC}) + B(\gamma_{BC})$$
$$= A(\gamma_1) + A(\gamma_2)$$

$A(\gamma_1) \geq A(\gamma_C)$ $\to$ QED

$A(\gamma_2) \geq A(\gamma_{ABC})$

Can also see: $S(AC) + S(BC) > S(A) + S(B)$



- Can get entanglement entropy of (1+1)-CFT

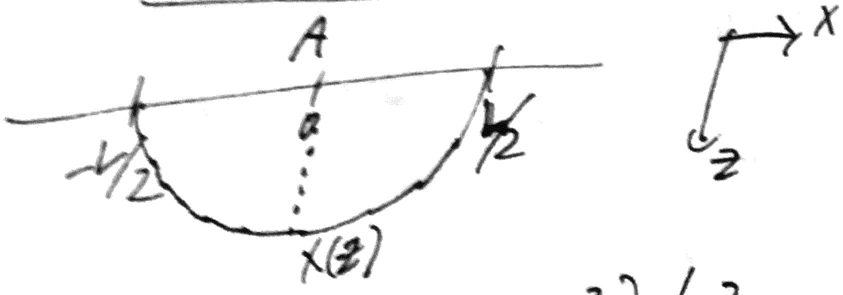
We've seen $CFT_4 \leftrightarrow AdS_5$ $\quad N^2 \sim 1/G_N$

can also construct $CFT_2 \leftrightarrow AdS_3$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + dx^2\right)$$

each CFT_2 is characterized by a central charge c

$$\boxed{c = \frac{3R}{2G_N}}$$



$$dl^2 = \frac{R^2}{z^2}\left((1 + x'(z)^2\right)dz^2 \Rightarrow S_A = \frac{1}{4\pi}\cdot 2 \cdot \int_0^{z_*} dz\, \frac{R}{z}\sqrt{1+x'^2}$$

$$x(z=0) = L/2$$

IG8 (From high school, know geodesic in hyperbolic space is semicircle half-circle: $x = \sqrt{\frac{L^2}{4} - z^2}$

$$S_A = \frac{1}{4G_N} 2R \frac{L}{2} \int_0^{L/2} \frac{dz}{z} \frac{1}{\sqrt{\frac{L^2}{4} - z^2}}$$

logarithmic UV divergence

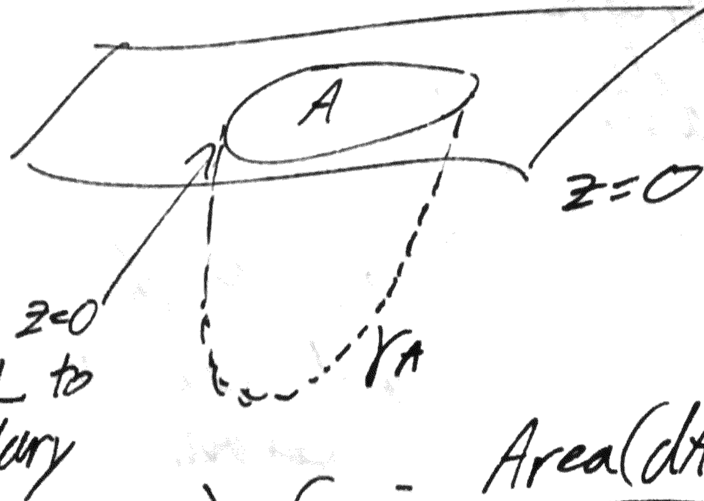
$\Downarrow$

change to ε

$$\rightarrow \frac{1}{3} \cdot \underbrace{\frac{3R}{2G_N}}_{c} \log \frac{L}{\varepsilon}$$

- At finite temperature, (**) is compatible with Beckenstein-Hawking formula for black hole entropy

- Area law for general dimensions:



near $z=0$ γ_A is $\perp$ to boundary

$$\Rightarrow S_A = \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-2}} + \ldots$$

$\uparrow$

Ryu-Takayagi lets us understand this leading order term

Final words

why should we expect entropy, the quantum information of a system, be related to the area of a region in some spacetime?

Ryu-Takayanagi formula implies:

spacetime $\longleftrightarrow$ geometrization of quantum entanglement

geometry $\longleftrightarrow$ quantum informat